An Alternative Approach to Deriving Rodrigues' Formula

Aki Anandam

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1 Introduction

This article outlines a method to achieving Rodrigues' Formula without using a series approach. Rodrigues' Formula states that the polynomial solution to the Legendre ODE,

$$\frac{d}{dx}(x^2-1)y' = l(l+1)y, \tag{1}$$

is

$$y = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.^1 \tag{2}$$

The standard approach to solving (1) is to let y be a power series and to find a recurrence relation between the coefficients of the series. One can then show that (2) satisfies such a relation. It is not immediately obvious, however, the manner in which repeated derivatives of $(x^2-1)^l$ appears in the solution. Such is the aim of this article. We wish to show directly how repeated derivatives of $(x^2-1)^l$ can appear and how one can uncover (2).

2 The initial idea

We will start by rewriting (1) into

$$y = \frac{1}{l(l+1)}\frac{d}{dx}(x^2-1)y'. \tag{3}$$

Then, we will differentiate both sides of (3):

$$y' = \frac{1}{l(l+1)}\frac{d^2}{dx^2}(x^2-1)y'. \tag{4}$$

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[1] The coefficient $\frac{1}{2^l l!}$ is simply to normalize the solution such that $y(1) = 1$, and any constant replacing it would yield a solution.

Then, we will substitute (4) into (3) to find

$$y = \frac{1}{l^2(l+1)^2}\frac{d}{dx}(x^2-1)\frac{d^2}{dx^2}(x^2-1)y'.$$

This process can be repeated and theoretically infinite number of times, so

$$l^n(l+1)^n y = \frac{d}{dx}(x^2-1)\frac{d^2}{dx^2}(x^2-1)\frac{d^2}{dx^2}(x^2-1)\frac{d^2}{dx^2}(x^2-1)...\frac{d^2}{dx^2}(x^2-1)y'$$

such that there are an n number of (x^2-1) terms. It is already clear how repeated derivatives and multiple powers of (x^2-1) are significant, but it is still not obvious how we can get exactly (2).

To get a better grasp on this, we will examine the first few values of n and see if there is any way we can extract certain solutions.

n	Equation
1	$(x^2-1)y^{(2)} + 2xy^{(1)} = l(l+1)y$
2	$(x^2-1)^2y^{(4)} + (8x^3-8x)y^{(3)} + (14x^2-6)y^{(2)} + 4xy^{(1)} = l^2(l+1)^2y$
3	$(x^2-1)^3y^{(6)} + (18x^5-36x^3+18x)y^{(5)} + (98x^4-124x^2+26)y^{(4)} + (184x^3-136x)y^{(3)} + (100x^2-36)y^{(2)} + 8xy^{(1)} = l^3(l+1)^3y$

Table 1: The different equations resulting from different values for n

where $y^{(n)}$ denotes the nth derivative of y. These equations get out of hand very quickly, but there are some things that can be extracted from them.

First, we will note that in their current state, each new equation is not actually saying anything the previous equations did not. Each equation is just every other equation written differently. Second, it is important to establish that by themselves, these equations can never yield (2). This is because the Legendre polynomials are not the only solutions to the Legendre ODE. There is a whole other set of solutions that can be represented with an infinite series. To reach (2), it is necessary to find an assumption that is true for all Legendre polynomials but not true for the other solutions. We will then apply the assumption to the equations to make it easily solvable for y. The way we will do this is to examine an equation of a similar form as (1), namely

$$\frac{d}{dx}(x^2)y' = l(l+1)y.$$

One may immediately notice that a solution to this differential equations is cx^l. From there it is possible to reason that because a solution to this equation is a polynomial of the lth degree, perhaps a solution to (1) is also a polynomial of the lth degree. This fact is obvious looking at (2), but we will be assuming we do not already know Rodrigues' formula when deriving it. In that case, one assumption we may make that is correct for these polynomials is

$$\frac{d^{l+1}}{dx^{l+1}}y = 0.$$

This is certainly true for all lth order polynomials, so perhaps it is true for y in (1). If we use this new assumption now and let l equal n, we see we can rewrite the $n = 1$ case as

$$2xy' = 2y.$$

Whenever we have a differential equation of the form

$$f(x)y^{(n)} = f^{(n)}y,$$

we have a solution

$$y = cf(x)$$

where c is an arbitrary constant. This case is true for our $n = 1, l = 1$ example, so

$$y_1 = 2cx$$

where y_k denotes the solution to (1) when $l = k$. Now, we did not necessarily have to use the $n = 1$ equation to find y_1. We could have done the same with our $n = 2$ equation, utilizing our assumption, to say

$$4xy_1' = 4y_1.$$

$$y_1 = 4cx.$$

This is identical to our $n = 1$ case. For simplicity and consistency, however, we will focus on the $n = l$ equation and those that came before it. Finding y_1 was easy because our assumption naturally applied, but what if we wanted to find y_2? Our $n = 1$ equation will not be useful because we cannot just apply the assumption, since the other cases include a $y^{(1)}$ term when all we want is a $y^{(2)}$ term. To get it so that we only have terms of $y^{(2)}$ and higher, one thing we can do is subtract two times the $n = 1$ equation from the $n = 2$ equation. This will yield

$$(x^2-1)^4y_2^{(4)} + (8x^3-8x)y_2^{(3)} + (12x^2-4)y_2^{(2)} = ((l^2(l+1)^2 - 2l(l+1))y_2.$$

Applying our assumption now leaves

$$(12x^2-4)y_2'' = 24y_2$$

which again, means

$$y_2 = 12x^2 - 4.$$

We will use a similar method of eliminating lower derivative terms of y to find y_3. We will start with our $n = 3$ equation. Then we will subtract eight times the $n = 2$ equation:

$$(x^2-1)^3y^{(6)} + (18x^5 - 36x^3 + 18x)y^{(5)} + (90x^4 - 108x^2 + 18)y^{(4)} +$$
$$(120x^3 - 72x)y^{(3)} + (-12x^2 - +12)y^{(2)} - 24xy^{(1)} = (l^3(l+1)^3 - 8l^2(l+1)^2)y.$$

We will then add twelve times the $n = 1$ equation, so

$$(x^2-1)^3y^{(6)} + (18x^5 - 36x^3 + 18x)y^{(5)} + (90x^4 - 108x^2 + 18)y^{(4)} +$$
$$(120x^3 - 72x)y^{(3)} = (l^3(l+1)^3 - 8l^2(l+1)^2 + 12l(l+1))y.$$

This is our equation to find y_3. We do a similar thing to find y_4 and subsequent values.

We will take a step back to analyze our method for finding solutions thus far. To find y_l, we analyze the equation $n = l$. We then subtract from it or add to it constant multiples of the equations of $n = l - 1$ and lower. We do this in such a manner that, on the left hand side, the lowest derivative of y is $y^{(l)}$. We will call such equations whose lowest derivative of y is $y^{(l)}$ *reduced* equations. It is not immediately obvious how one can determine which constants for which equations to subtract or add such that we get a reduced equation, and the constants from the examples given were only found through analyzing each individual equation which is a process that cannot be generalized. With $l = 2$, we had to subtract two times the $n = 1$ equation from the $n = 2$ equation. With $l = 3$, we had to subtract 8 times the $n = 2$ equation and add twelve times the $n = 1$ equation to the $n = 3$ equation. The aim of the next section is to find a natural and clear pattern to these these operations for reaching reduced equations. We will then show then when applying the generalized method for reaching a reduced equation, the function multiplied to the $y^{(l)}$ term is $d^l/dx^l(x^2-1)^l$. If we can show that the right hand side of the reduced equation equals $d^l/dx^l[d^l/dx^l(x^2-1)^l]$, then we will achieved our goal of reached the formula.

3 A generalization of the reduction method

In this section, we will find a general pattern to find the reduced equations for each and all values of l. First, we will start by defining the operator, $\mathcal{D}$, such that

$$\mathcal{D} = \frac{d}{dx}(x^2-1)\frac{d}{dx}.$$

We can then rewrite (1) into

$$\mathcal{D}y_l = l(l+1)y_l \tag{5}$$

Now, we will examine a theoretical reduced equation of the form

$$f_{2l}(x)y^{(2l)} + f_{2l-1}(x)y^{(2l-1)} + ...f_l(x)y^{(l)} = (l^l(l+1)^l - ...)y.$$

where $f_k(x)$ denotes the function multiplied by the y_k term. Because of our assumption, we know

$$y_l = f_l(x).$$

Therefore, we can use (5) to say

$$\mathcal{D}f_l(x) = l(l+1)f_l(x)$$

or

$$(\mathcal{D} - l(l+1))f_l(x) = 0.$$

This means that if we apply the operator $(D - l(l+1))$ to the equation $n = l$, then the $y^{(l)}$ term will vanish. This is what we are looking for, because the resulting left hand side of the new equation will only include terms of $y^{(l+1)}$ or higher. There is no way in which lower derivatives of y can appear since we are at all times either differentiating or multplying by $(x^2 - 1)$ the derivatives of y. We have found a pattern with how we achieve our reduce polynomials. To make the $y^{(1)}$ term in equation $n = 1$ disappear, apply the operator $\mathcal{D}$. This will give us a reduced polynomial and our solution for y_2. To make the $y^{(2)}$ term in the second degree reduced equation disappear, apply $\mathcal{D} - 2$. To make the $Y^{(3)}$ term in the third degree reudced equation, apply $\mathcal{D} - 6$. We see this method is naturally recursive.

We have essentially established a method to find y_l for any integer l. We start with

$$y_l = y_l.$$

Then, we apply $\mathcal{D}$, utilizing (5) to rewrite the right hand side:

$$\mathcal{D}y_l = l(l+1)y_l$$

Then, we apply $\mathcal{D} - 2$:

$$(\mathcal{D}(\mathcal{D} - 2))y_l = (l^2(l+1)^2 - 2l(l+1))y_l.$$

Then, we apply $\mathcal{D} - 6$:

$$(\mathcal{D}(\mathcal{D} - 2)(\mathcal{D} - 6))y_l = (l^3(l+1)^3 - 8l^2(l+1)^2 + 12l(l+1))y_l.$$

We keep repeating this , each time applying $\mathcal{D} - n(n+1)$ to equation n until l is one less than n because we do not want the $y^{(l)}$ to vanish. Doing this, we end up with

$$(\mathcal{D}(\mathcal{D} - 2)(\mathcal{D} - 6)..(\mathcal{D} - l(l-1))y_l = (l^l(l+1)^l - ...)y_l. \tag{6}$$

The function multiplied to the $y^{(l)}$ term is our solution so long as its lth derivative is the coefficient of y_l on the right side of the equation. What we must do now is show the $y^{(l)}$ term on the left side is $\frac{d^l}{dx^l}(x^2-1)^l$ and the coefficient of y_l on the right is $(2l)!$ which is the lth derivative of $\frac{d^l}{dx^l}(x^2-1)^l$. We will start with the former.

4 Reaching Rodrigues' Formula

The aim of this section is to show the function multiplied by the $y^{(l)}$ term in (6) is $\frac{d^l}{dx^l}(x^2-1)^l$.

We will first examine in further detail the left hand side of (6),

$$(\mathcal{D}(\mathcal{D}-2)(\mathcal{D}-6)..(\mathcal{D}-l(l-1))y_l.$$

We will examine the first few iterations of this in Table 2.

l	Reduce Left Hand Side
1	$(x^2-1)y^{(2)}+2xy^{(1)}$
2	$(x^2-1)^2y^{(4)}+(8x^3-8x)y^{(3)}+(12x^2-4)y^{(2)}+2xy^{(1)}$
3	$(x^2-1)^3y^{(6)}+(18x^5-36x^3+18x)y^{(5)}+(90x^4-108x^2+18)y^{(4)}+(120x^3-72x)y^{(3)}$
4	$(x^2-1)^4y^{(8)}+(32x^7-96x^5+96x^3-32x)y^{(7)}+(336x^6-720x^4+432x^2-48)y^{(6)}+(1344x^5-1920x^3+576x)y^{(5)}+(1680x^4-1440x^2+144)y^{(4)}$

Table 2: The different left hand sides of reduced equations

One thing that is clear is that the highest derivative term is always y^{2l} as a result of the fact that we are applying at most two differential operators for each value of l. The coefficient of this term is always $(x^2-1)^l$, because the y_l portion in the product rule will consume all the derivatives leaving just the (x^2-1) portion of $\mathcal{D}$ left over. There is another interesting pattern one may see develop. For clarity, we will adopt the convention that f_a^b represents the function multiplied to $y^{(b)}$ in the left hand side of the reduced equation $n=a$. In this way, one may see the pattern that it appears

$$f_l^{2l-u}=\frac{n-u+1}{u}\frac{d}{dx}f_l^{2l-u+1} \tag{7}$$

where u represents are arbitrary integer less than or equal to l.

For example, we see that

$$f_3^4=90x^4-108x^2+18,$$

and

Figure 1: Diagram of reduced functions and the relation between one iteration and the next

$$f_3^5 = 18x^5 - 36x^3 + 18x.$$

Since f_3^4 is the same as f_3^{2*3-2}, that would mean our u value for (7) would be 2. We can then use this specific example to verify (7), because indeed

$$f_3^4 = \frac{3-2+1}{2}\frac{d}{dx}f_3^5.$$

In fact, one could use any reduced equation they wish including those not listed in Table 2 to verify (7). Now, we will show (7) is true for all values of l and u, and once we have that, we can reach $\frac{d^l}{dx^l}(x^2-1)^l$. To do so, we will utilize induction. Induction lends itself nicely to our method because it is naturally recursive, where upon each iteration, we apply the operator $(D - l(l+1))$.

First, we will begin with our base. We will show that (7) is true when u equals 1. To do so, we will have to make use of the fact

$$f_a^b = (x^2-1)f_{a-1}^{b-2} + \frac{d}{dx}(x^2-1)f_{a-1}^{b-1} + (x^2-1)\frac{d}{dx}f_{a-1}^{b-1} + \frac{d}{dx}(x^2-1)\frac{d}{dx}f_{a-1}^{b} - a(a-1)f_{a-1}^{b}. \tag{8}$$

This can be derived from analyzing which functions, when applying $D - a(a-1)$ can lead to f_a^b. There are only at most three potential terms. There is the f_{a-1}^{b-2} term which swallows derivatives, allowing the y_{a-1} to take them, to become f_a^b. Then there is the f_{a-1}^{b-1} term which has its pick from either of the two derivatives in $\mathcal{D}$, giving the other derivative to y_{a-1} or the (x^2-1) to become f_a^b. Finally there is the $f-a-1^b$ term which, combined with (x^2-1) must swallow up all the derivatives or will be multiplied by $a(a-1)$. Both routes for the last option are viable, so they are both included. The reader is encourage to work through the reasoning above to understand the conclusion.

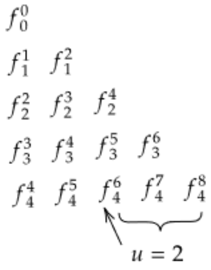

Figure 2: u represents the distance of a function from the rightmost function

For our case, we are dealing with u equal to 1. The last two terms are always zero, because we get a case of $f_{l-1}^{2(l-1)-1+2}$. $2l-1$ exceeds $2(l-1)$, so this must vanish, which can be seen in Figure 1 as there is no f_3^7 or f_4^9 or any other function in the position up and to the right of f_l^{2l-1}. With this, we rewrite the equation as

$$f_l^{2l-1} = (x^2-1)f_{l-1}^{2(l-1)-1} + \frac{d}{dx}(x^2-1)f_{l-1}^{2(l-1)} + (x^2-1)\frac{d}{dx}f_{l-1}^{2(l-1)}.$$

Now, our goal is to show

$$f_l^{2l-1} = l\frac{d}{dx}f_l^{2l},$$

which is just (7) when u equals 1.

To do so, we can use (8) or just reasoning to say

$$f_l^{2l} = (x^2-1)^l$$

Now, to finish this, we will utilize induction once more. We are trying to show (7) holds when u equals 1 for our equation of l. We will assume it holds for the equation of $l-1$, or

$$f_{l-1}^{2(l-1)-1} = (l-1)\frac{d}{dx}f_{l-1}^{2l-1}.$$

Since this holds for the our base that f_1^1 equals the derivative of f_1^2, we may build off of this to show the whole case of $u=1$ satisfied (7). This then means

$$f_l^{2l-1} = (l-1)(x^2-1)\frac{d}{dx}f_{l-1}^{2(l-1)} + \frac{d}{dx}(x^2-1)f_{l-1}^{2(l-1)} + (x^2-1)\frac{d}{dx}f_{l-1}^{2(l-1)},$$

or

$$f_l^{2l-1} = 2x(l-1)^2(x^2-1)^{l-1} + 2xl(x^2-1)^{l-1} + 2x(l-1)(x^2-1)^{l-1}.$$

$$f_l^{2l-1} = 2xl^2(x^2-1)^{l-1} = l\frac{d}{dx}f_l^{2l-1}.$$

We have now proved the base of our induction for the case of $u = 1$. We will proceed to show that if (7) holds for $u-1$, then it holds for u.

So, we assume (7) holds for $u-1$. This implies

$$f_l^{2l-k} = \frac{l!}{k!(l-k)!}\frac{d^k}{dx^k}(x^2-1)^l \tag{9}$$

where k is less than u. One may recognize the coefficient of the derivatives as the choose operator from combinatorics. Our goal will then be to show

$$f_l^{2l-u} = \frac{l!}{u!(l-u)!}\frac{d^u}{dx^u}(x^2-1)^l. \tag{10}$$

We can also restate (9) as

$$f_{l-1}^{2(l-1)-k} = \frac{(l-1)!}{k!(l-k-1)!}\frac{d^k}{dx^k}(x^2-1)^{l-1}. \tag{11}$$

We can use (8) to then say

$$\begin{aligned} f_l^{2l-u} =& (x^2-1)f_{l-1}^{2(l-1)-u} + \frac{(l-1)!}{(u-1)!(l-u)!}\left[\frac{d}{dx}(x^2-1)\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1}\right. \\ &\left. + (x^2-1)\frac{d^u}{dx^u}(x^2-1)^{(}l+1)\right] + \frac{(l-1)!}{(u-2)!(l-u+1)!}\frac{d}{dx}(x^2-1)\frac{d^u}{dx^u}(x^2-1)^{l-1} \\ & - \frac{l(l-1)(l-1)!}{(u-2)!(l-u+1)!}. \end{aligned}$$

We have the final piece to reaching (10) will utilize induction, for one final time, to say

$$f_{l-1}^{2(l-1)-u} = \frac{(l-1)!}{u!(l-u+1)!}\frac{d^u}{dx^u}(x^2-1)^{l-1}.$$

We show that if it holds for the case of $l-1$, then it will hold for the case of l. Since this is true for the base (as we'll see in a minute), then we can prove for the whole value of u. We would have then proved in (7) holds for $(u-1)$, then it holds for u, and because we have already proved the base of $u = 1$ above, then it will be true for all possibilities. We are essentially utilizing induction within induction. The reason for this is because the way we have formulated our reduced equations is recursive where we get each reduced equation from the previous reduced equation, so naturally, proving properties about such equations is easier with induction which is naturally recursive rather than other more explicit means. Now that we have all the information we need, we may say

$$\begin{aligned} f_l^{2l-u} =& \frac{l!}{u!(l-u)!}\Bigg[(x^2-1)\frac{d^u}{dx^u}(x^2-1)^{l-1} + \frac{u}{l}\frac{d}{dx}(x^2-1)\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} \\ &+ \frac{u(u-1)}{l(l-u+1)}\frac{d}{dx}(x^2-1)\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} \\ &- \frac{u(l-1)(u-1)}{l-u+1}\frac{d^{u-2}}{dx^{u-2}}(x^2-1)^{l-1}\Bigg]. \end{aligned}$$

Now, before we may go any further, we must develop an expansion for

$$(x^2-1)\tfrac{d^u}{dx^u}(x^2-1)^{l-1}$$

and

$$\tfrac{d}{dx}(x^2-1)\tfrac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1}.$$

This will allow us to combine the different terms, ultimately bringing us closer to (10).

We will start out generally, first developing an expression for

$$g(x)\tfrac{d^k}{dx^k}h(x)$$

which is just

$$g(x)\tfrac{d}{dx}\tfrac{d}{dx}\tfrac{d}{dx}...\tfrac{d}{dx}h(x).$$

We can then use the reverse of the product rule to write this as

$$\tfrac{d}{dx}g(x)\tfrac{d^{k-1}}{dx^{k-1}}h(x) - \tfrac{d}{dx}g(x)\tfrac{d^{k-1}}{dx^{k-1}}h(x).$$

We will use the reverse product rule once more to get

$$\tfrac{d^2}{dx^2}g(x)\tfrac{d^{k-2}}{dx^{k-2}}h(x) - \tfrac{d}{dx}[\tfrac{d}{dx}g(x)]\tfrac{d^{k-2}}{dx^{k-2}}h(x) - \tfrac{d}{dx}g(x)\tfrac{d^{k-1}}{dx^{k-1}}h(x).$$

And once more...

$$\tfrac{d^2}{dx^2}g(x)\tfrac{d^{k-2}}{dx^{k-2}}h(x) - \tfrac{d^2}{dx^2}[\tfrac{d}{dx}g(x)]\tfrac{d^{k-3}}{dx^{k-3}}h(x) - \tfrac{d}{dx}[\tfrac{d}{dx}g(x)]\tfrac{d^{k-2}}{dx^{k-2}}h(x) - \\ \tfrac{d}{dx}g(x)\tfrac{d^{k-1}}{dx^{k-1}}h(x).$$

A pattern begins to emerge, and we see that

$$g(x)\frac{d^k}{dx^k}h(x) = \frac{d^n}{dx^n}g(x)\frac{d^{k-n}}{dx^{k-n}}h(x) - \sum_{i=0}^{n-1}\frac{d^i}{dx^i}[\frac{d}{dx}g(x)]\frac{d^{k-i-1}}{k-i-1}h(x)$$

for any integer value of $n$less than or equal to k. If we then let n equal k, we see

$$g(x)\frac{d^k}{dx^k}h(x) = \frac{d^k}{dx^k}g(x)h(x) - \sum_{i=0}^{k-1}\frac{d^i}{dx^i}[\frac{d}{dx}g(x)]\frac{d^{k-i-1}}{k-i-1}h(x). \tag{12}$$

We can then differentiate both sides of (12) to get

$$\frac{d}{dx}g(x)\frac{d^k}{dx^k}h(x) = \frac{d^{k+1}}{dx^{k+1}}g(x)h(x) - \sum_{i=0}^{k}\frac{d^i}{dx^i}[\frac{d}{dx}g(x)]\frac{d^{k-i}}{k-i}h(x). \tag{13}$$

We can apply (12) and (13) to our specific cases to say

$$(x^2-1)\frac{d^u}{dx^u}(x^2-1)^{l-1} = \frac{d^u}{dx^u}(x^2-1)^l - \sum_{i=0}^{u-1}\frac{d^i}{dx^i}2x\frac{d^{u-i-1}}{dx^{u-i-1}}(x^2-1)^{l-1}$$

$$\frac{d}{dx}(x^2-1)\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} = \frac{d^u}{dx^u}(x^2-1)^l - \sum_{i=1}^{u-1}\frac{d^i}{dx^i}2x\frac{d^{u-i-1}}{dx^{u-i-1}}(x^2-1)^{l-1}.$$

Now, as a result of the fact $\frac{d^3}{dx^3}(x^2-1)$ equals zero, most of the expanded terms in $\frac{d^i}{dx^i}2x\frac{d^{u-i-1}}{dx^{u-i-1}}(x^2-1)^{l-1}$ vanish, leaving only

$$\frac{d^i}{dx^i}2x\frac{d^{u-i-1}}{dx^{u-i-1}}(x^2-1)^{l-1} = 2x\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} + 2k\frac{d^{u-2}}{dx^{u-2}}(x^2-1)^{n-1}. \tag{14}$$

We can therefore use (14) to evaluate the sums, leaving finally

$$\frac{d^u}{dx^u}(x^2-1)^{l-1} = \frac{d^u}{dx^u}(x^2-1)^l - 2ux\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} - u(u-1)\frac{d^{u-2}}{dx^{u-2}}(x^2-1)^{l-1} \tag{15}$$

$$\frac{d}{dx}(x^2-1)\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} = \frac{d^u}{dx^u}(x^2-1)^l - 2(u-1)x\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} - u(u-1)\frac{d^{u-2}}{dx^{u-2}}(x^2-1)^{l-1}. \tag{16}$$

Utilizing (15) and (16), we push through the algebra to get

$$f_l^{2l-u} = \frac{l!}{u!(l-u)!(l-u+1)}\Bigg[(l+1)\frac{d^u}{dx^u}(x^2-1)^l - 2uxn\frac{d^{u-1}}{dx^{u-1}}(x^1-)^{l-1} - (2un-2nu^2)\frac{d^{u-2}}{dx^{u-2}}(x^2-1)^{l-1}\Bigg].$$

We then expand $\frac{d^u}{dx^u}(x^2-1)^l$, utilizing (14), and divide through by $l-u+1$ to get

$$f_l^{2l-u} = \frac{l!}{u!(l-u)!}\left[2xl\frac{d^{u-1}}{dx^{u-1}}(x^2-1)^{l-1} + 2l(u-1)\frac{d^{u-2}}{dx^{u-2}}(x^2-1)^{l-1}\right].$$

We can they use the reverse of (14) to finally say

$$f_l^{2l-u} = \frac{l!}{u!(l-u)!}\frac{d^u}{dx^u}(x^2-1)^l, \tag{17}$$

which is exactly what we set out to show! This is just another form of (7) which is the pattern we we trying to verify. The last order of business to is show that this holds for our inductive bases. This is not that difficult because is we choose our base to be f_l^{2l-l-1}, then indeed

$$f_l^{2l-l-1} = \frac{l-l-1+1}{l+1}\frac{d}{dx}f_l^{2l-l-1+1},$$

because it equals zero. Now that we have (17), we can let u equal l to say

$$f_l^l = \frac{d^l}{dx^l}(x^2-1)^l.$$

All we must do now is verify the right hand side agrees with the left (the lth derivative of $\frac{d^l}{dx^l}(x^2-1)^l$ equals the coefficient of y_l on the right).

5 The Final step

Because of (5) and (6), we know the right hand side of a reduce equation is

$$\mathcal{D}(\mathcal{D}-2)(\mathcal{D}-6)...(\mathcal{D}-l(l-1))y_l = y_l\prod_{n=1}^{l}(l(l+1)-n(n-1)).$$

We wish to show that this equals the lth derivative of $\frac{d^l}{dx^l}(x^2-1)^l$, or $(2l)!$. To do so, we will let

$$k = l-n,$$

so

$$\mathcal{D}(\mathcal{D}-2)(\mathcal{D}-6)...(\mathcal{D}-l(l-1))y_l = y_l\prod_{k=0}^{l-1}(l(l+1)-(l-k)(l-k-1).$$

In other words,

$$\mathcal{D}(\mathcal{D}-2)(\mathcal{D}-6)...(\mathcal{D}-l(l-1))y_l = y_l \prod_{k=0}^{l-1}(2l-k)(k+1) = (2l)!.$$

We have finally proved the last step which means that when solving Legendre's ODE and assuming $\frac{d^{l+1}}{dx^{l+1}}y_l = 0$, we get

$$(\frac{d^l}{dx^l}(x^2-1)^l)y_l^{(l)} = y_l(2l)!,$$

which therefore has the solution

$$y_l = c\frac{d^l}{dx^l}(x^2-1)^l.$$

Finally, if we let our constant, c, equal $\frac{1}{2^l l!}$, then we get *normalized* solutions where $y_l(1) = 1$, so

$$y_l = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l. \tag{18}$$

(18) is Rodrigues' Formula which has many applications in fields such as electromagnetism.