

Torques and One Dimensional Rigid Bodies

Aki Anandam

February 2021

1 Introduction

The inspiration for this paper arose from this problem:

Suppose you have a one dimensional rod of length l whose mass is evenly distributed. You are to give it a certain impulse I , for a period of time Δt at the point on the rod P which is a distance d from the center of mass. [A] hopefully makes clear this system.

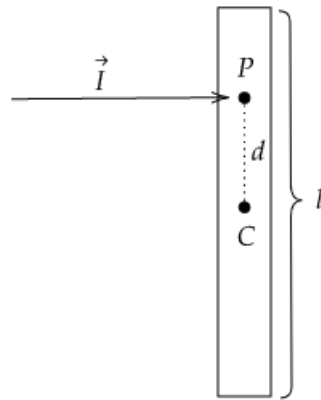


Figure [A]: A diagram of the system we're working with

After the duration of time Δt , the rod will be rotating with some angular velocity ω . The problem is simply to find the value for ω . We will approach this problem in the reference frame of the point, P , instead of the point, C , which is usually chosen. Before we do that, I'll briefly address notation. In this paper, I'll be working with the notation that $r_{pi}^{\vec{}}$ would indicate the position vector from the point, P , to some hypothetical point, I . Also, $a_{pi}^{\vec{}}$ would indicate the acceleration P sees I have, and $v_{pi}^{\vec{}}$ is the velocity P sees I have. With this in mind, we are able to proceed to analyzing the angular momentum and torques of this system about P .

2 Angular Momentum

The angular momentum of a point is simply

$$\vec{L} = \vec{r} \times \vec{p}$$

where \vec{r} is the position vector from the reference frame to the point and \vec{p} is the momentum the reference frame sees the point have. Our reference frame is P, and we will be dealing with a system of points. In such cases, the total angular momentum is just the sum of the angular momentum's of each small portion of the rod. Thus

$$\vec{L}_p = \sum_{i=1}^N \vec{r}_{pi} \times \vec{p}_{pi}$$

where i represents some arbitrary small portion of the rod, N represents the total number of these small portions, and \vec{p}_{pi} is simply the momentum P observes the small portion, i , has. We want the magnitude of the angular momentum, but we want to account for the sign. To do this, we'll adopt the coordinate system where a vector pointing towards the reader and out of the paper is positive, and a vector pointing down and away from the reader is negative. When we do this and expand \vec{p}_{pi} , we get

$$L_p = \sum_{i=1}^N m_i (\vec{r}_{pi} \times \vec{v}_{pi}). \quad (1)$$

\vec{v}_{pi} is just the velocity P sees i have. Now, we will translate \vec{r}_{pi} and \vec{v}_{pi} into vectors in the center of mass frame to simplify this problem. We utilize these two facts

$$\vec{r}_{pi} = \vec{r}_{pc} + \vec{r}_{ci} \quad (2)$$

$$\vec{v}_{pi} = \vec{v}_{pc} + \vec{v}_{ci} \quad (3)$$

Notice, if we differentiate (3) with respect to time, t , we get

$$\vec{a}_{pi} = \vec{a}_{pc} + \vec{a}_{ci} \quad (4)$$

Now, we'll substitute the values from (2) and (3) into one, distribute, and get

$$L_p = \sum_{i=1}^N m_i (\vec{r}_{pc} \times \vec{v}_{pc} + \vec{r}_{pc} \times \vec{v}_{ci} + \vec{r}_{ci} \times \vec{v}_{pc} + \vec{r}_{ci} \times \vec{v}_{ci}). \quad (5)$$

Since the mass is evenly distributed about the rod, we can say

$$\sum_{i=1}^N m_i \vec{r}_{ci} = 0. \quad (6)$$

This allows us to get rid of the third cross product because $v_{pc}^{\vec{}}$ is not dependent on i , so we can take it out of the sum, and we'll be left with $v_{pc}^{\vec{}}$ crossed with (6), which is just 0. Also, the final cross product is just the angular momentum in C 's reference frame. When we account for these facts, we arrive at

$$L_p = L_c + \sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times v_{pc}^{\vec{}} + r_{pc}^{\vec{}} \times v_{ci}^{\vec{}}).$$

Now, if we look back to [A], we see that $v_{pc}^{\vec{}}$ is just ωd , and $r_{pc}^{\vec{}}$ is d , when we evaluate the cross product and account for our sign convention and the fact that the sum of all m_i is just the total mass, M , we can reduce this to

$$L_p = L_c - M\omega d^2 + \sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times v_{ci}^{\vec{}}).$$

Since $r_{pc}^{\vec{}}$ isn't dependent on i , we can factor it out of the sum. Doing this yields

$$L_p = L_c - M\omega d^2 + r_{pc}^{\vec{}} \times \sum_{i=1}^N m_i v_{ci}^{\vec{}}.$$

$m_i v_{ci}^{\vec{}}$ is just the momentum of the small portion in C 's reference frame, so when we sum it all up, we get

$$L_p = L_c - M\omega d^2 + r_{pc}^{\vec{}} \times \vec{P}_c$$

where \vec{P}_c is just the total momentum of the rod in C 's reference frame. Now, \vec{P}_c is just zero. This is because the mass is evenly distributed, so the momentum of one point on the rod is exactly cancelled out by the momentum on the other end of the rod. Since each point has exactly one point that is opposite it, the momenta will evenly cancel. [B] hopefully makes this fact clear.

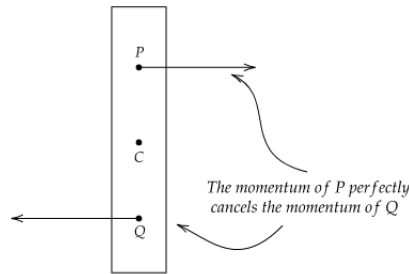


Figure [B]: The momenta perfectly cancel, so the net momentum is zero.

Since the total momentum in C 's reference frame is zero, any vector crossed with it will also be zero, so

$$L_p = L_c + M\omega d^2.$$

We can expand L_c , making sure to keep our sign convention because $r_{ci}^{\vec{}} \times v_{ci}^{\vec{}}$ is negative. We finally end up with

$$L_p = -\left(\frac{1}{12}\omega Ml^2 + M\omega d^2\right) \quad (7)$$

It should make sense that the angular momentum is negative because the rod is rotating clockwise, and in our coordinate system clockwise rotation is negative while counter clockwise rotation is positive. This is analogous to the fact that positive curl is counter clockwise while negative curl is clockwise. Granted, this does seem like a long and hard way to show that the angular momentum about P is just the momentum of inertia about P multiplied by ω , but it can be useful to see the true derivation of this fact instead of just taking it for granted.

Now that we've found the angular momentum about P when the rod is rotating counter clockwise with angular velocity ω , we'll analyze the torque induced on the rod in P 's reference frame as we apply some impulse, I , to P .

3 Torque

The analysis of the torque will follow similar reasoning as the analysis of the angular momentum. The torque of one point about another point is

$$\tau = \vec{r} \times \vec{F}$$

where \vec{F} is the force our reference point sees applied to the point in question. Like with angular momentum, the total torque on the rod is just the sum of torques on small portions of the rod, so

$$\tau_p = \sum_{i=1}^N r_{pi}^{\vec{}} \times F_{pi}^{\vec{}}$$

We can expand $F_{pi}^{\vec{}}$ to get

$$\tau_p = \sum_{i=1}^N m_i (r_{pi}^{\vec{}} \times a_{pi}^{\vec{}}). \quad (8)$$

We'll use (2) and (4) to say

$$\tau_p = \sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times a_{pc}^{\vec{}} + r_{pc}^{\vec{}} \times a_{ci}^{\vec{}} + r_{ci}^{\vec{}} \times a_{pc}^{\vec{}} + r_{ci}^{\vec{}} \times a_{ci}^{\vec{}}). \quad (9)$$

Like we did in the angular momentum section, we can cancel the third cross product because of (6). This yields

$$\tau_p = \sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times a_{pc}^{\vec{}} + r_{pc}^{\vec{}} \times a_{ci}^{\vec{}} + r_{ci}^{\vec{}} \times a_{ci}^{\vec{}}). \quad (10)$$

Recall from the previous section that

$$\sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times v_{pc}^{\vec{}}) = -M\omega d^2.$$

We can take the derivative of both sides with respect to t , utilizing the fact that $c_{pc}^{\vec{}} \times v_{pc}^{\vec{}}$ is zero, to say

$$\sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times a_{pc}^{\vec{}}) = -M\alpha d^2. \quad (11)$$

where α is the angular acceleration. Inputting (11) into (10) gives

$$\tau_p = -M\alpha d^2 + \sum_{i=1}^N m_i (r_{pc}^{\vec{}} \times a_{ci}^{\vec{}} + r_{ci}^{\vec{}} \times a_{ci}^{\vec{}}). \quad (12)$$

Now is where things become interesting. We are dealing with the acceleration of each small segment in C 's reference frame. It is important to realize that **every** point on the rod is accelerating in C 's reference frame.

To make this more clear, we will imagine the system from C 's point of view. We can treat applying our impulse like a collision. Before the collision, every single particle on the rod was not moving relative to C . After the collision, **every** single particle began rotating about C . **Every** single particle began moving with velocity ωr . If a particle goes from having 0 velocity to have some non zero velocity, it must have accelerated. Since every particle did that, **every** particle must have accelerated. Since the particles toward the end of the rod end up moving faster than the particles moving toward the center of the rod, and all particles started with no velocity, the particles toward the end of the rod had a larger acceleration than the particles toward the center of the rod. [C] and [D] hopefully makes these statements more clear.

Now we return to (12). When we look at the first cross product, we see that we can factor out $r_{pc}^{\vec{}}$. We can rewrite (12) as

$$\tau_p = -M\alpha d^2 + r_{pc}^{\vec{}} \times \sum_{i=1}^N m_i a_{ci}^{\vec{}} + \sum_{i=1}^N m_i (r_{ci}^{\vec{}} \times a_{ci}^{\vec{}}).$$

The mass is evenly distributed, so we can disregard its value when considering the second sum. From [D], it is clear that the sum of all accelerations is zero, so the second sum vanishes. This leaves us with

$$\tau_p = -M\alpha d^2 + \sum_{i=1}^N m_i (r_{ci}^{\vec{}} \times a_{ci}^{\vec{}}).$$

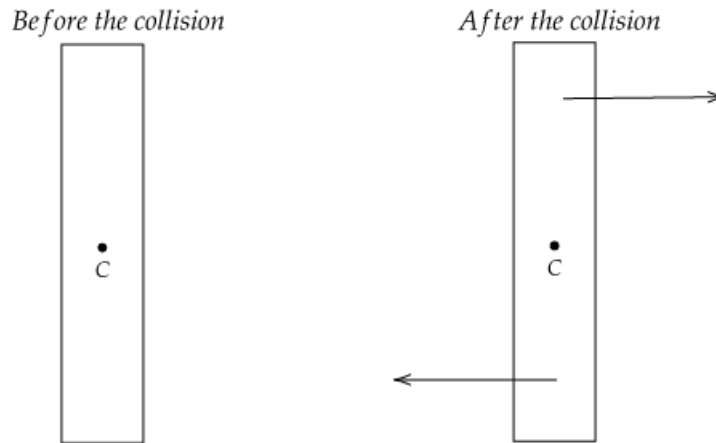


Figure [C]: Every particle goes from not moving to moving, so every particle accelerated

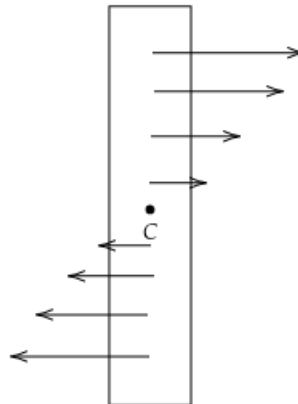


Figure [D]: A diagram of the accelerations of every point on the rod

We are left with the final sum, which is simply the torque in C 's reference frame. One may want to just say that the torque in C 's reference frame is simply the force multiplied by the distance from the center to the point the force is applied to, but why is that? It's clear when we're dealing with point particles, but why does this definition of torque still hold when we are dealing with a rod? After all, [D] clearly shows that every point on the rod is accelerating, so we must account for every single point and not just the point, P . We are going to explore this idea in more depth and show why the torque still is the force

multiplied by the distance to the center even when we have a rod where every point is accelerating.

4 Reasoning for Torque

When we hit the rod, every point begins to accelerate. This implies that a force causes each point to accelerate horizontally, so there must be a torque on each point. This means that if we want to find the total torque on the rod, we have to account for the accelerations of every single point on the rod and not just the point where we hit the rod, P . This raises the question, if every single point on the rod has a torque, why is it that when we take the sum of every torque, we still arrive at Fd , where d is just the distances from P to the center of mass. After all, how does P even show up if every single point is accelerating, and more importantly, why does the torque still remain Fd ? First, we have to recognize that the reason every single point on the rod accelerates horizontally in the first place is because of inter molecular forces between each particle. These are just electromagnetic forces that can be represented by Coulomb's Law. The key idea that is true for every force in the universe, thanks to Newton's third law, is that these forces come in pairs, equal and opposite. The force that attracts point A to point B must also attract point B back to point A . We will examine two systems, one in which inter molecular forces are present, and another in which they are not. We will represent the rod as a one dimensional stream of particles, depicted in [E].

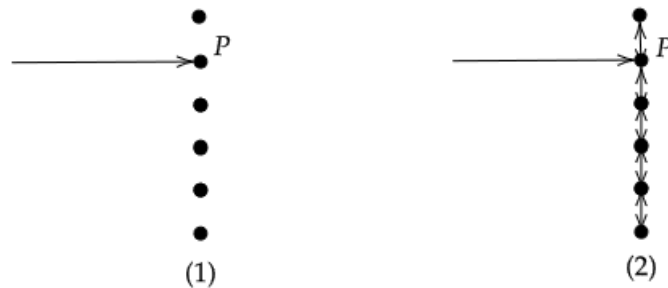


Figure [E]: Two systems with and without inter molecular forces

In [E], (1) represents a system where P is hit and there aren't any inter molecular forces, and (2) represents a system where P is hit and there are inter molecular forces. Note, in this representation, the inter molecular forces are only

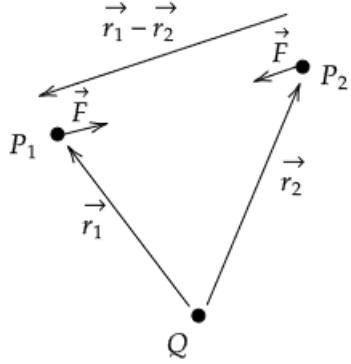


Figure [F]: Calculating the torque of two particles with internal forces

between neighboring points. This isn't needed as we'll see shortly, and was only made so for the sake of simplicity in drawing. In reality, every point attracts every other point, although not equally strongly. Now, calculating the torque in (1) is easy because *now*, there really is one force and one accelerations. Only P is going to move from the hit, not any other point. So, we know for certain that that the net torque in (1) is just Fd . What about the net torque in (2)? We know the net torque is the sum of all torques, but there are so many forces to account for! It seems like this would be a complete headache. In fact, it's not, and this is solely because our Newton's third law we discussed above. First, we will account for the force that acts on P , giving us the same force, Fd . It's important to note that although every point is accelerating, the force only acts on P . The only forces acting on every other point are the inter molecular forces. Now, what about the torques caused by the inter molecular forces? Surely we'd have to include that? Actually, we don't. The remarkable thing is that they all simply cancel to zero. To see this clearly, we'll think about two particles, although this generalizes extremely easily to multiple pairs of particles.

We see in [F] that we have two particles with attracted forces holding them together. We see that the net torque in Q 's frame is

$$\tau_Q = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times -\vec{F} \quad (13)$$

Note that the diagram doesn't represent this but the force acting on P_2 is technically $-F$ because it is in the opposite direction. We can then say that

$$\tau_Q = \vec{r}_1 \times \vec{F} - \vec{r}_2 \times \vec{F}$$

$$\tau_Q = (\vec{r}_1 - \vec{r}_2) \times \vec{F}.$$

Since F is a force that always points from one particle to the other, $\vec{r}_1 - \vec{r}_2$ will always be co-linear with F , so the cross product is just zero! That means that the net torque is zero. This generalizes very easily to system with many particles, because we can apply the exact same logic to every single pair of points. This also works even if the particles aren't neighboring which explains why we can include every single particle attraction. This is really why physics professors say that angular momentum is conserved when there is no net external torque, because internal torque will always add to zero regardless of where the two particles are. This may seem counter intuitive given that the two particles could have different positions and therefore would have different torques, but the key idea is that the forces are equal and opposite which allows us to say the torque's can. This is why internal torques have no effect on the angular momentum-because they always sum to zero. Applying this to our rod case with (2), we see that the inter molecular forces are all internal forces and thus, they form internal torques. This allows us to disregard them because we know they will all sum to zero and just say that the net torque is Fd .

5 A quick wrap up and solution

To summarize the previous sections,

$$L_p = - \left(\frac{1}{12} \omega M l^2 + M \omega d^2 \right) \quad (14)$$

and

$$\tau_p = -M \alpha d^2 - Fd.$$

Remember, these are both negative because the rotation is clockwise. Any mention of them being positive above is only discussing their magnitudes. Since

$$\tau_p \Delta t = L_p$$

and

$$\alpha \Delta t = \omega,$$

$$- \left(\frac{1}{12} \omega M l^2 + M \omega d^2 \right) = -M \omega d^2 - F \Delta t d$$

If $F \Delta t$ is an impulse, I , then we can solve for ω to get

$$\omega = \frac{12 I d}{M l^2}.$$