

# On the Particle Model of Elasticity

Aki Anandam

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## 1 Introduction

We will be exploring the notion of elasticity and Hooke's Law in the view of a particle model. To begin, we'll start with Hooke's law and formula for the force. If you have some spring of an equilibrium length,  $l$ , and you displace it by some amount,  $x$ , then the restorative spring force that will act is given by

$$F = -kx \tag{1}$$

where  $k$  is the spring constant unique to each spring. This is depicted in [A].

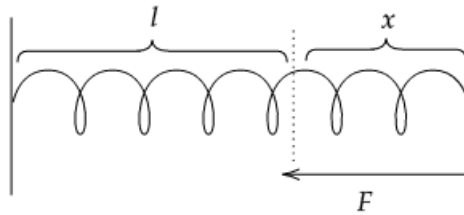


Figure [A]: A spring displaced by  $x$

We want to give the reasoning for why the force is linearly proportional to the displacement. To do so, we will understand what actually causes the spring to return back to its equilibrium state. The force spring force is really the electrostatic force, which is represented by Coulomb's Law:

$$F = \frac{\kappa q_1 q_2}{r^2}. \tag{2}$$

$\kappa$  is Coulomb's constant,  $q_1$  and  $q_2$  are the charges of the particles in question, and  $r$  is the distance between the two particles. Normally, we would multiply by the unitary  $\hat{r}$ , but because we are going to be dealing with a one dimensional case, we can let the sign take care of the direction. The way a spring works is that there are two electrostatic forces acting on the spring, an attractive one and a repulsive one. The repulsive one acts over smaller distances while the attractive one works over larger distances. At the equilibrium point, both of these forces are equal, so the spring doesn't move. However, when you displace the spring by stretching it, then the attractive force will become stronger than the repulsive force.<sup>1</sup> Likewise, if you squish the spring, then the repulsive force will become stronger. This is why the spring always pushes against your pull and always pulls against your push. To make these ideas more clear, we will examine just two particles. Later, we'll justify why this holds for a one dimensional stream of particles like a spring.

The way we'll set it up so that the attractive force works over longer distances is to say that the product of the charges is greater in the a attractive force, while the distance is smaller in the repulsive force. The disparity in the distances will become more and more prevalent as the two particles become closer and closer together which is what gives the repulsive force priority in this domain.

Here is how we will set up the attractive and repulsive forces: we will call the attractive force  $A$  and the repulsive force  $R$ . We know (2) governs both of these forces. We'll call  $\kappa q_1 q_2$  in  $A$ ,  $g$ , and we'll call  $\kappa q_1 q_2$  in  $R$ ,  $h$ . Then, we'll say  $r$  in the repulsive force is  $l+x$  where  $l$  is still the distance during equilibrium and  $x$  is still the displacement. Then, we'll say the distance in the attractive force is  $l+x+\Delta r$ .  $\Delta r$  represents the disparity in the distances between the two forces. All of this allows us to say

$$A = -\frac{g}{(l+x+\Delta r)^2} \quad (3)$$

and

$$R = \frac{h}{(l+x)^2}. \quad (4)$$

Our sign convention will be such that the increasing  $x$  direction is to the right. All of these decisions may seem unmotivated, but they are all reasonable, and we will explain why that is so. First, we will examine [B], which displays two particles a distance  $l$  apart.

For the remainder of the paper, we will imagine point  $A$  is fixed and only point  $B$  can move. This is analogous to have a spring where one end is attached to a wall or table. Now, we know from everyday experience that if we pull  $B$  away from  $A$ , then there will be a force trying to bring  $B$  back to equilibrium. Likewise, if we bring  $B$  closer to  $A$ , then there will be a force trying to push  $B$  away back to equilibrium. One force can't simultaneously push  $B$  and pull it. This implies that there must be two forces acting on  $B$ . One that pushes it

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<sup>1</sup>It's not so much the attractive force somehow becomes stronger than it was originally because that can't be since the distance is getting larger. Rather, it is that the repulsive force is getting weaker faster than the attractive force

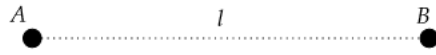


Figure [B]: Two particles a distance  $l$  apart

and one that pulls it. This also works out nicely for our interpretation of our equilibrium point. It is simply the point where the pulling force and the pushing force are equal. This justifies our assumption of the existence of two forces. Now, we must justify why the attractive force has both a greater product of charges and a greater distance. We can think back to our experience of springs. When we increase the distance, we feel  $B$  trying to return to  $A$ . This means that over larger distances, the attractive force is greater than the repulsive force. Conversely, when we push  $B$  to  $A$ ,  $B$  wants to resist the push. This means that over smaller distances, the attractive force is lower than the repulsive force. One helpful thing to think about is what these attractive and repulsive forces actually are.

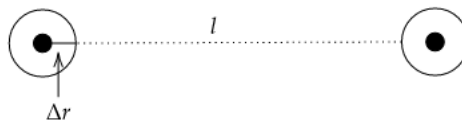


Figure [C]: Two particles with orbiting electrons

If we imagine a substance like water, the force that holds molecules together are hydrogen bonds. Hydrogen bonds are just the electrostatic force. Now,

because there is a cloud of electrons orbiting, those electrons are going to be slightly closer to the electrons of other molecules compared to the nucleus. The electrons repel each other and are attracted to the nucleus. The repulsion of the electrons represents the repulsive force, and attraction between the electrons and the nuclei represent the attractive force. Because the electrons are closer together, the repulsive force must be divided by a smaller distance compared to the attractive force. The difference in distance, labeled  $\Delta r$ , is simply the distance in one particle from its nucleus to its electrons depicted in *C*. However, since the nucleus has more charge density compared to the electrons, the product of charges is going to be greater. This is the reasoning for why the product of charges in the attractive force is greater than the product of charges in the repulsive force. All of this explains why a spring always seems to return to its equilibrium point (provided it's not sufficiently stretched).

Armed with (3) and (4), we now try to figure out what the force must be on *B* when it is displaced by some amount  $x$ . To have a positive displacement, we will move *B* to the right, depicted in [D].



Figure [D]: *B* displaced by  $x$

The total force acting on *B* is just the sum of the attractive and the repulsive force, so

$$F = \frac{h}{(l+x)^2} - \frac{g}{(l+x+\Delta r)^2}. \quad (5)$$

Now, we can expand (5) using a Taylor expansion. We will approximate  $F$ , but cutting off the Taylor expansion of both terms after the second term. The reason we are allowed to do this is because we are going to assume  $x$  is sufficiently smaller than  $l$ . This is not just a reasonable assumption. It is the very basis for Hooke's Law! Indeed, the law doesn't hold if  $x$  is many times larger than  $l$ , because at that point, the spring would get deformed and could even break. This approximation is actually exactly what we need to do to satisfy the

conditions of Hooke's Law. So, when we Taylor expand this and approximate, we get

$$F = \frac{h}{l^2} - \frac{2hx}{l^3} - \left( \frac{g}{(l + \Delta r)^2} - \frac{2gx}{(l + \Delta r)^3} \right). \quad (6)$$

When we simplify, we get

$$F = \frac{h}{l^2} - \frac{g}{(l + \Delta r)^2} + x \left( \frac{2g}{(l + \Delta r)^3} - \frac{2h}{l^3} \right). \quad (7)$$

We then reexamine [B]. As we've already stated, at the equilibrium point, the attractive force and the repulsive force add to zero. Since  $x$  is zero there, we can say

$$\frac{h}{l^2} - \frac{g}{(l + \Delta r)^2} = 0. \quad (8)$$

We can then substitute (8) into (7) to arrive at

$$F = x \left( \frac{2g}{(l + \Delta r)^3} - \frac{2h}{l^3} \right). \quad (9)$$

which is a spectacular result because it is exactly what we wanted to prove! The coefficient of  $x$  is simply some constant that is unique to each spring. This is clear, because  $g$ ,  $h$ , and probably  $\Delta r$  vary with each spring, which is what allows us to call the whole thing a spring constant.

One slight caveat is that when we inspect (1), we see that  $k$  must be positive and must have a negative coefficient. This is simply a result of the fact that  $F$  is a restoring force and is trying to bring  $B$  back in the negative  $x$  direction. A more precise equation for  $F$  would be

$$F = - \left( \frac{2h}{l^3} - \frac{2g}{(l + \Delta r)^3} \right) \cdot x. \quad (10)$$

This raises the question, is the coefficient of  $-x$  always positive? Indeed it is so. We will promptly show why

$$\frac{2h}{l^3} - \frac{2g}{(l + \Delta r)^3} > 0$$

or in other words

$$\frac{2h}{l^3} > \frac{2g}{(l + \Delta r)^3}. \quad (11)$$

To do this, we return to (8). When we solve for  $g$ , we get

$$g = \frac{h(l + \Delta r)^2}{l^2}. \quad (12)$$

We can then use (12) to say that

$$\frac{2g}{(l + \Delta r)^3} = \frac{2h}{l^2(1 + \Delta r)} = \frac{2h}{l^3 + l^2\Delta r}. \quad (13)$$

Now it is clearly that always

$$\frac{2h}{l^3} > \frac{2h}{l^3 + l^2\Delta r}$$

regardless of how small  $\Delta r$  is.

We have now given strong reasoning for Hooke's Law for the force of a spring. It is important to note that we can express  $l$  in terms of  $h$ ,  $g$ , and  $\Delta r$ . Those quantities should be viewed as inherit to the spring, and  $l$  can simply be determined from them. What we must do now is show how this holds for multiple points.

We then proceed to examine [E], in which we have a system of many particles. These particles are in all respects identical to those we've been dealing with already. The forces are the same, and  $\Delta r$  is the same. The total equilibrium length between them is  $l$ .

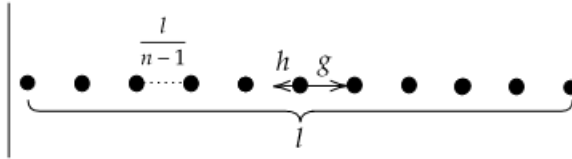


Figure [E]: A system of  $n$  particles with equilibrium length  $l$

If there are  $n$  particles, and the total length from start to finish is  $l$ , then each particle is a distance  $\frac{l}{n-1}$  away from its neighbors. Because  $l$  is the equilibrium distance for the whole system,  $\frac{l}{n-1}$  is the equilibrium distance for each pair of neighboring points. That means that the repulsive and attractive forces equal when the particles are separated by  $\frac{l}{n-1}$ . Here,  $\frac{l}{n-1}$  takes the place of  $l$  in our two particle example. Equations like (8) still hold so long as you make the right substitution. Now, we will displace the final point by moving it to the right by  $x$ . This will cause the total length to become  $l + x$ , just like in our two particle case. Note, in order for the system to become stable, the net force on every

particle must be zero. The only way this can happen is if every point is evenly spaced. This is a result of the fact that  $g$  and  $h$  are the same for every point. If they weren't evenly spaced, then the attractive or repulsive forces might be stronger on some particles rather than others, which will yield some net force which will destabilize the system. Since every particle remains evenly spaced, the new distance between neighboring particles becomes  $\frac{l+x}{n-1}$  as seen in [F].

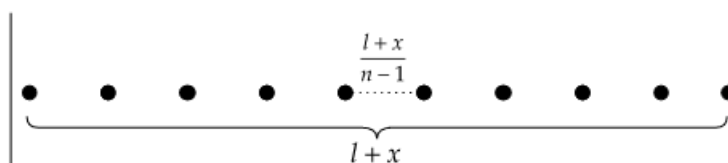


Figure [F]: The system extended by  $x$

$\frac{l+x}{n-1}$  now takes the place of  $x$  in our two particle example. It is important to note that all of the equations applicable to our two particle example apply so long as we substitute properly. In fact, there really is no difference between the actual mechanics of our system of many points. In a sense, our system of many points is just a large collection of systems of two points. And because (9) still applies and  $\frac{l+x}{n-1}$  is still linear with  $x$ , we are able to say that Hooke's law easily translates into a system of many particles. One important thing to note is that we are talking strictly on the force the particle on the right end feels, not the forces the other particles feel. In fact, at the instant you let go, the only particle feeling a net force is that rightmost particle. Every other particle will feel a force to the left, but they will also feel an equal force to the right. The only potential exception could be the leftmost point, but we've already established that is fixed in place by something like a connection to a wall. In the next instant, of course, this will not be the case. The rightmost particle would have moved such that now the particle to its left will not feel as strong attraction to it and will promptly move to the left. This will cause a chain effect with all the particles until everything begins to move to the left. They will all stop when they reach the leftmost particle, because there is no more room to move. Then, they will slowly bunch together, and the repulsive force will start to take precedent. In that case, the particles will start to repel each other and move back to the right. All of these motions explain the harmonic oscillations spring undergo when you

extend them and then suspend them freely.

## 2 Some things to note

When you increase the length of the spring, that increases the number of particles in the spring, not the distance between them. Also, the force we apply only acts on the final point, and the inter molecular forces pull the rest of the spring. That means that our force stops having an effect when the last point reaches its maximum extension, which means every point reaches its maximum extension. Note that the maximum extension for the last point is independent of the number of points behind it. That means that the more particles, the more maximum extensions, and the longer the overall displacement. Since the length is dependent on the number of particles, the length increases the maximum displacement in a linear fashion, justifying

$$\Delta l \propto l.$$

Also, the surface area of the spring can be thought of as the number of these once dimensional strands bunched together. Since the force will have to spread out evenly throughout each strand, the surface area is inversely proportional to the displacement, justifying

$$\Delta l \propto \frac{1}{A}.$$

When we also include the fact that

$$\Delta l \propto F$$

which we showed above, we can say

$$\Delta l \propto \frac{lF}{A}.$$

Therefore

$$\frac{F}{A} = Y \frac{\Delta l}{l}.$$

where  $Y$  is Young's Modulus.  $Y$  can be thought of as how stiff the rod is, and is the factor that dictates how little a rod is affected (strain) by some stress. The higher  $Y$ , the lower the strain.