Elliptical Orbits and Kepler's Laws

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1 Introduction

Kepler's laws govern the orbits of planets and other celestial bodies. Kepler gave three laws in terms of the planets orbiting the sun:

Law 1 *All planets orbit the sun in ellipses with the sun at one focus*

Law 2 *Planets will sweep out equal areas in equal times.*

Law 3 *The period of the planet's orbit squared is proportional to the length of the major axis of its orbit cubed.*

[1] illustrates Law 2.

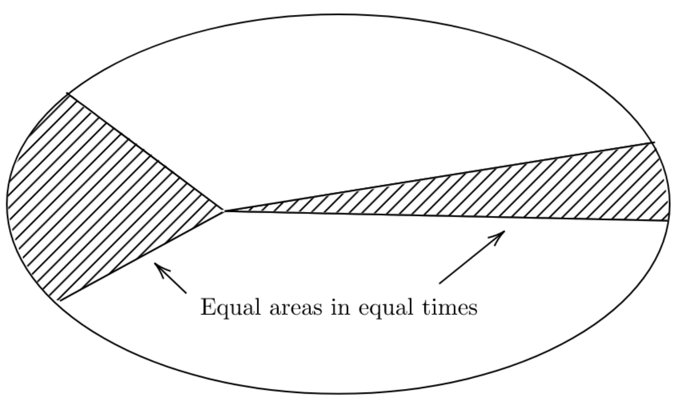

Figure [1]: Equal areas in equal times

Our goal is to derive these laws, starting with the first.

2 The First Law

Law 1 *All Planets orbit the sun in ellipses*

There are many approaches when it comes to these laws, but they all start from the same place – Newton's law of gravitation:

$$F_g = \frac{-Gm_1m_2}{r^2} \tag{1}$$

Where G is the gravitation constant, m_1 and m_2 are the masses of the two objects, and r is the distance between them. Note, this equation only describes the magnitude of the force. A more precise definition would be

$$\vec{F_g} = \frac{-Gm_1m_2}{r^2}\hat{r} \tag{2}$$

where $\vec{F}$ is now a vector, and $\hat{r}$ is the unit vector in the direction of the radius, $\vec{r}$ which is simply

$$\hat{r} = \frac{\vec{r}}{r}$$

Now we will make some clarifications. For the rest of the paper, m_2 is going to be orbiting m_1, and $\vec{r}$ is the vector from m_1 to m_2.

In this derivation, we will be using complex numbers.

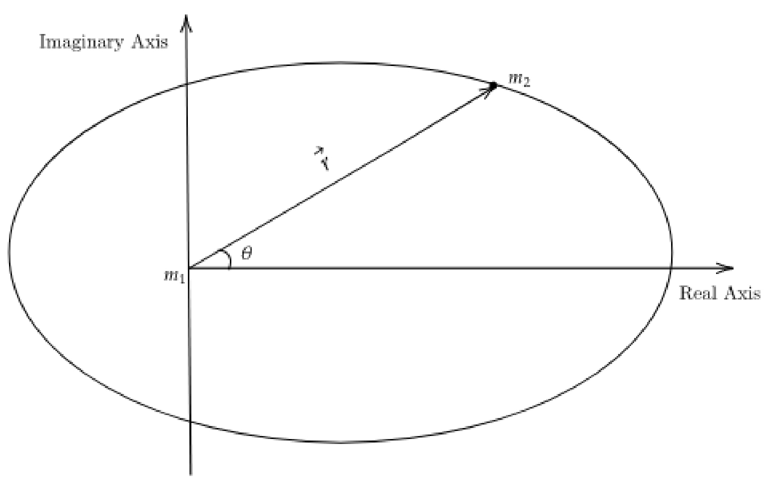

Figure [2]: An illustration of the system we're working with

m_1 is centered at the origin, and m_2 is at the point with modulus r and argument θ

Here, both r and θ functions of the time variable, t. The reason we are using polar coordinates in the complex plane instead of vectors in $\mathbf{R}^2$ is because the formula for the force of gravity lends itself nicely to polar coordinates because it is opposite the direction of the radius, and the complex plane is the best way to represent polar coordinates. If we let $p(t)$ be m_2's position as a function of time [1], then we get

$$p(t) = r(t)e^{i\theta(t)} \tag{3}$$

We know by Newton's second law that

$$F = ma$$

or

$$a = \frac{F}{m}$$

Since the only force acting on m_2 is gravity, and acceleration is just the second derivative of the position, $p(t)$, we can say

$$\frac{-Gm_1}{r(t)^2}e^{i\theta} = p''(t) \tag{4}$$

Since $\hat{r}$ is the unit vector in the direction of r, and $e^{i\theta}$ is a unit complex number in the direction of $p(t)$, we can substitue $e^{i\theta}$ for $\hat{r}$. When we evaluate $p''(t)$ using the equation above, we get

$$p''(t) = e^{i\theta}\left(r'' + 2ir'\omega + ir\alpha - r\omega^2\right) \tag{5}$$

where ω is the derivative of θ with respect to time and α is the derivative of ω with respect to time.

When we equate the two terms and cancel out $e^{i\theta}$, we get

$$r'' + i(2r'\omega + r\alpha) - r\omega^2 = \frac{-Gm_1}{r^2} \tag{6}$$

Now we will examine the torque of this system with reference point at m_1. The torque is simply

$$\tau_{m_1} = \vec{r} \times \vec{F}$$

Where $\vec{r}$ is the position vector and $\vec{F}$ is the force vector. We have already established that the force in is in the opposite direction of the position vector. Therefore, the cross product between the position vector and the force is zero, so the torque on this system is zero.

The torque is simply the derivative of the angular momentum with respect to time. The angular momentum, L, of m_2 about m_1 is given by

$$L_{m_1} = I_{m_1}\omega \tag{7}$$

[1] We are going to assume that $m_1 >>> m_2$ so that the motion of m_1 is negligible, and we can assume that m_1 is fixed at the origin.

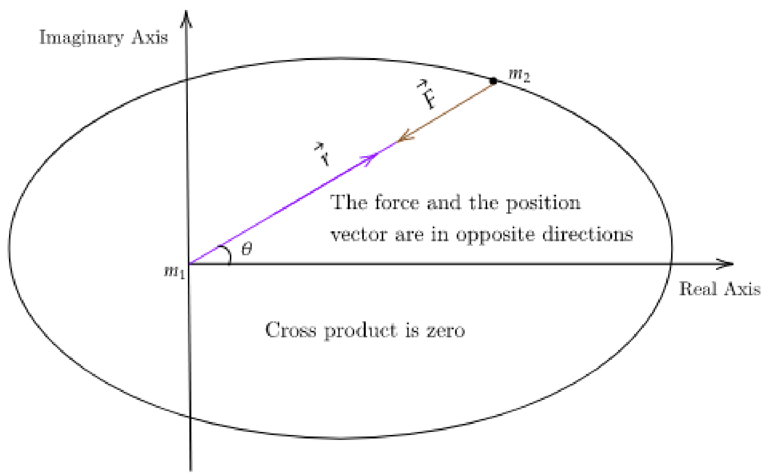

Figure [3]: The torque on m_2 in m_1's reference frame is zero

I_{m_1} is just the moment of inertia of m_2 about m_1 which is m_2r^2 where r is the same distance we've been dealing with. Inputting, we get,

$$L_{m_1} = m_2r^2\omega \tag{8}$$

Since the torque is $\frac{dL}{dt}$, we can say

$$\tau = \frac{d}{dt}m_2r^2\omega = m_2(2rr'\omega + r^2\alpha)$$

We have already established that the torque, τ, is zero, so

$$m_2(2rr'\omega + r^2\alpha) = 0$$

We can cancel m_2 and r to get

$$2r'\omega + r\alpha = 0 \tag{9}$$

We can apply (9) back to (6) to get

$$r'' - r\omega^2 = \frac{-Gm_1}{r^2} \tag{10}$$

This simplified our equation greatly. In this is differential equation, however, both r and θ are functions of time, and their derivatives appear. Our goal for this section is simply to show that the planets orbit in ellipses. Knowing that, it is irrelevant how fast they move or what their angular velocity is. We simply care about the general shape of the orbit. So, we will translate this differential equation into a differential equation where r is a function of θ. This will give us a second order differential equation that we can solve. We will make the following changes

$$r'' = \frac{d}{dt}\frac{dr}{dt} = \omega\frac{d}{d\theta}\frac{dr}{dt} \tag{11}$$

Since

$$\frac{dr}{dt} = \frac{dr}{d\theta}\omega \tag{12}$$

We say

$$r'' = \omega\frac{d}{d\theta}\frac{dr}{d\theta}\omega = \frac{d^2r}{d\theta^2}\omega^2 + \alpha\frac{dr}{d\theta} \tag{13}$$

Therefore,

$$r''\omega^2 + r'\alpha - r\omega^2 = \frac{-Gm_1}{r^2} \tag{14}$$

It is important to recognize that now r' and r'' are functions of θ, not t. We used to the chain rule to get (11) and (12). This allowed us to translate the differential equation into one where r is a function of θ

We know from (9) that

$$2r'(t)\omega + r\alpha = 0$$

It is important to note that here $r'(t)$ is as a function of t. In order to translate it to a function of θ, we must replace it with $r'\omega$. When we do this and solve for α, we get

$$\alpha = \frac{-2r'\omega^2}{r}$$

Inputting (15) into (14) yields

$$r''\omega^2 - \frac{2r'^2\omega^2}{r} - r\omega^2 = \frac{-Gm_1}{r^2} \tag{15}$$

We can divide by ω, and, using (8), we say

$$\omega^2 = \frac{L^2}{m_2^2r^4} \tag{16}$$

so

$$r'' - \frac{2r'^2}{r} - r = \frac{-Gm_1m_2^2r^2}{L^2} \tag{17}$$

This may look considerably harder than our original differential equation, but it is actually much easier. Why? Because now we only have one variable and its derivatives. Before, both r and θ were functions of time, and we had successive derivatives of both. Using the fact that angular momentum was constant, we were able to eliminate ω into simply constants and r. We then just used the chain rule to rewrite r' and r'' into functions of θ.

We'll now proceed to solve. First, let's clean things up and bring all the r's to the left

$$\frac{r''}{r^2} - \frac{2r'^2}{r^3} - \frac{1}{r} = \frac{-Gm_1m_2^2}{L^2}$$

Notice that all of the r's are in the denominator. To find our solution, we will make use of a substitution that utilizes this fact.

$$r = \frac{1}{u}$$

So

$$r' = -\frac{u'}{u^2}$$

$$r'' = \frac{2u'^2}{u^3} - \frac{u''}{u^2}$$

This is a very helpful substitution because it allows us to cancel that pesky r' term. After going through the algebra, we arrive at

$$u'' + u = \frac{Gm_1m_2^2}{L^2}$$

This is a great result because it is easily solvable. The general solution to this is

$$u = Acos(\theta - \phi) + \frac{Gm_1m_2^2}{L^2}$$

So

$$r = \frac{1}{Acos(\theta - \phi) + \frac{Gm_1m_2^2}{L^2}} \tag{18}$$

Which is the polar form of an ellipse, exactly as was to be shown. The mass being orbited, m_1 sits at the origin which is a focus of the ellipse, showing that the planets orbit the sun in ellipses with the sun at one focus.

3 The Second Law

Law 2 *Planets will sweep out equal areas in equal times.*

This is a simple application of the conservation of angular momentum.

We know that a small segment of area of a polar curve, dA, is given by

$$dA = \frac{1}{2}r^2 d\theta$$

This can easily be found if we approximate the area segment to a triangle with height r and base $rd\theta$. By the chain rule, we get

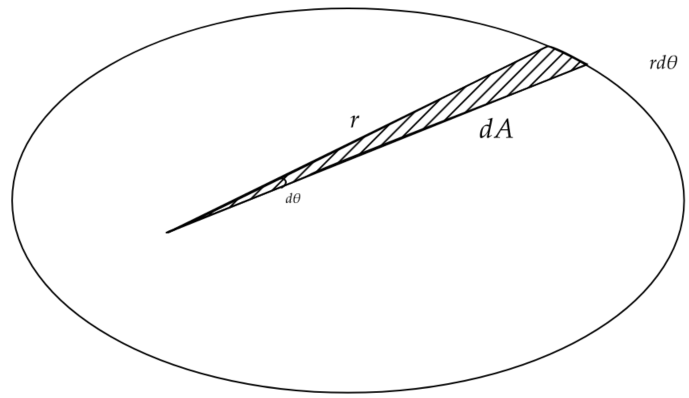

Figure [4]: Finding dA

$$dA = \frac{1}{2} r^2 \omega dt$$

We then divide by dt which is something mathematicians can't do but physicists can

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega$$

Using (8) again, we get

$$\frac{dA}{dt} = \frac{L}{2m_2}$$

We've already established that L is constant, so the whole right hand side is just a bunch of constants. We have just shown that $\frac{dA}{dt}$ is constant which means that the area is only proportional to the duration of time as was to be shown.

$$\frac{dA}{dt} = C$$

$$A = C\Delta t = \frac{L\Delta t}{2m_2} \tag{19}$$

4 The Third Law

Law 3 *The period of the planet's orbit squared is proportional to the major axis of it's orbit cubed.*

If we take (19) and let Δt be the orbital period, T, then the area spanned is going to be the entire area of the ellipse which is just πab where a and b are the major and minor axis, respectively

$$\pi ab = \frac{LT}{2m_2}$$

Squaring both sides, we get

$$\pi^2 a^2 b^2 = \frac{L^2 T^2}{4m_2^2}$$

Referring back to (18), we'll make the substitution:

$$e = \frac{Gm_1 m_2^2}{L^2} \tag{20}$$

to simplify things so

$$r = \frac{1}{Acos(\theta - \phi) + e} \tag{21}$$

Now to find a, and b, we consult [5].

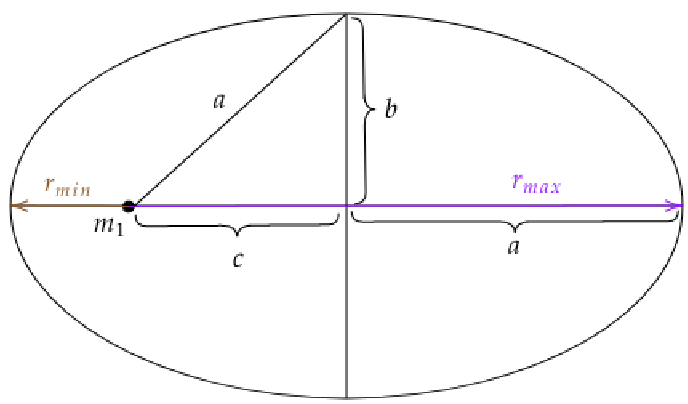

Figure [5]: Finding a and b from r

c is now the distance from the center of the ellipse to the focus, m_1, and r_{min} and r_{max} are the minimum and maximum values for r, respectively. We see that

$$r_{min} + r_{max} = 2a \tag{22}$$

and we also see that

$$c = a - r_{min} \tag{23}$$

and

$$b^2 = a^2 - c^2 \tag{24}$$

So, if we can find r_{min} and r_{max}, we will have a, b, and c (although we only needed b in terms of a).

Consulting (21), we see that the minimum value for r is when the denominator is maximized. Since e is a constant, the maximum value for the denominator is when $Acos(\theta - \phi)$ is maximized. The highest value the cosine graph can attain is 1, so

$$r_{min} = \frac{1}{e + A}$$

Similarly, the maximum value for r is when the denominator is minimized. This happens when the cosine graph reaches it's lowest point at -1, so

$$r_{max} = \frac{1}{e - A}$$

Using (22), we get that

$$a = \frac{e}{e^2 - A^2}$$

thus

$$c = \frac{A}{e^2 - A^2}$$

and so

$$b = \frac{1}{\sqrt{e^2 - A^2}} = \sqrt{\frac{a}{e}}$$

We now have b in terms of a, so we can go back to (40) to say

$$\frac{\pi^2 a^3}{e} = \frac{L^2 T^2}{4m_2^2}$$

from (20), we know that

$$L^2 = \frac{G m_1 m_2^2}{e}$$

so

$$\frac{\pi^2 a^3}{e} = \frac{G m_1 T^2}{4e}$$

and finally

$$T^2 = \frac{4\pi^2 a^3}{Gm_1} \tag{25}$$

which is what we wanted to show. The period of an orbit squared is proportional to it's major axis cubed.