Notes of Electrostatic Potentials

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1 Introduction

The aim of this article is to add my notes and understanding on the topic of Electrostatic Potentials. To my knowledge, most if not all of the information discussed has not been said before, especially not in [1] which this article is intended to be supplementary to. As a result, much of the information discussed in [1] will be glossed over or entirely omitted to avoid redundancy.

2 Laplace's Equation

The basis of chapter 3 is on the idea of finding the electric potential of a given charge configuration instead of finding the field, because the former is often easier to find. For area's of no charge, the electric potential, V, satisfies

$$\nabla^2 V = 0. \tag{1}$$

This is Laplace's equation, and [1] gives good physical intuition for its solutions, such as the distribution of heat in a room. The reason the solutions to (1) are not just lines or planes is that, in the two dimensional case for example, all that is needed is that $\frac{d^2V}{dx^2}$ is negative $\frac{d^2V}{dy^2}$. Planes have both these values set to zero, but so long as they are opposite, there is more freedom for what they may equal. Solutions (1) may have saddle points since $\frac{d^2V}{dx^2}$ could still equal $\frac{d^2V}{dy^2}$.

3 Uniqueness Theorems

This is a fundamental subject to potentials, because the fact that there exists one and only one solution to (1) for a given set of boundary conditions implies that any method one uses to find a solution to (1) that satisfies the boundary conditions must be there value of V. [1] does a good job of proving these uniqueness theorems. It is easy to think that physically there can only be one value for the potential, there cannot be some superposition between two simultaneous values of V (and by that same token there *must* be some value for V, which is the existence part of the existence and uniqueness theorem).

4 Method of Images

The method of images is one of the many examples in which the uniqueness theorems can be used to easily find solutions to (1). It's aim is to recreate a separate charge configuration with the same values at the boundary as the configuration for which one is trying to field the potential. Because of the uniqueness theorem, the two configurations must have the same potential. The method of images often involves a charge situated near a conducting surface, and the method details adding a separate "test charge" with opposite the real charge and in the position of a reflection of the charge across the surface of the conducting material. One note is that things being grounded means that they are connected to a theoretically infinite supply of electrons, and because many often like to call the Earth, which is typically used for grounding, potential 0, this is essentially equivalent to just defining a boundary condition of V equalling 0 for the area that is grounded. One note for the method of images is that the cancellation must always be in pairs, there is no other way to perfectly cancel the potential along an entire surface using any other combination besides pairs. It is important that one never adds test charges in the area over which they are trying to find the potential, because then (1) would not hold. The overarching idea behind the method of images is to find a separate charge configuration whose potential you know, which satisfies the same boundary conditions are the configuration in question. By the uniqueness theorems, the two configurations must therefore have the same potential.

5 Multipole Expansions

Multipole expansion is just like a Taylor series for potentials, and because it is centered at the origin, it can be thought of as a Mclaurin series. This centering at the origin is significant because, just like with Taylor series, shifting where it is centered about changes the expansion and approximation. Figure 1 details the diagram of a dipole moment which is discussed in [1]. The general idea is that when the index in the integral points to positive charges and away from negative ones, and when there is a configuration like in Figure 1, this results in a net vector point towards the positive charges. The resulting field on specific point is seen in Figure 2 as just the dot product between the dipole moment and the vector to the point, which makes sense given the closer, positive charges in Figure 2 would repel the point away.

What's important to note is in the formula for the multipole expansion given in [1], θ represents two different values, the angle between the $d\tau$ charge index in the integral, and the angle for the spherical coordinates of the point in question.

References

[1] David J. Griffiths, Introduction to Electrodynamics, Prentice Hall, 1989

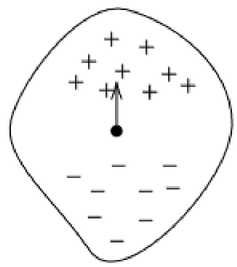

Figure 1: A diagram of a dipole moment

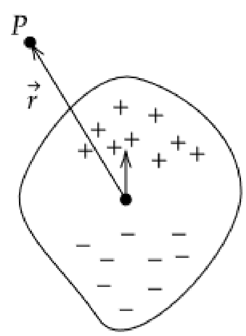

Figure 2: The field as the dot product of the position vector and dipole moment