On the Contour Integral of $1/z$

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1 Introduction

For any closed curve C enclosing the origin, we have the contour integral

$$\oint_C \frac{1}{z} dz = 2\pi i. \tag{1}$$

This result is foundational to the analysis of residues and winding numbers. We know from Cauchy's theorem that had the function we were integrating been analytic in the enclosed region, then the closed contour integral would have been guaranteed to be zero. The function $1/z$, however, is not analytic, since it blows up at the origin, which is what allows the contour integral to yield an nonzero value. It is tempting, then, to conclude that it is the "blowing up" at the origin which causes a nonzero result in (1). However, consider

$$\oint_C \frac{1}{z^2} dz = 0. \tag{2}$$

The function $1/z^2$ also has a singularity at the origin, but nevertheless the closed contour integral vanishes. More generally, if we plot the value of the class of contour integrals $\oint_C \frac{1}{z^k}$ as a function of k, we get the plot in Figure 1.

Of course, we have to be precise about fractional values of k since fractional exponents of z are multi valued and may require branch cuts, but this is not relevant for our discussion. We can suffice to consider only integer values of k, and we clearly see something special must be happening at $k = 1$. It clearly has nothing to do with the fact that the function blows up at the origin, since that's true for $k = 2, 3$, and so forth. The purpose of this article is to illustrate intuitively exactly what is special about the case $k = 1$ that leads to a nonzero contour integral, with a focus on visualization.

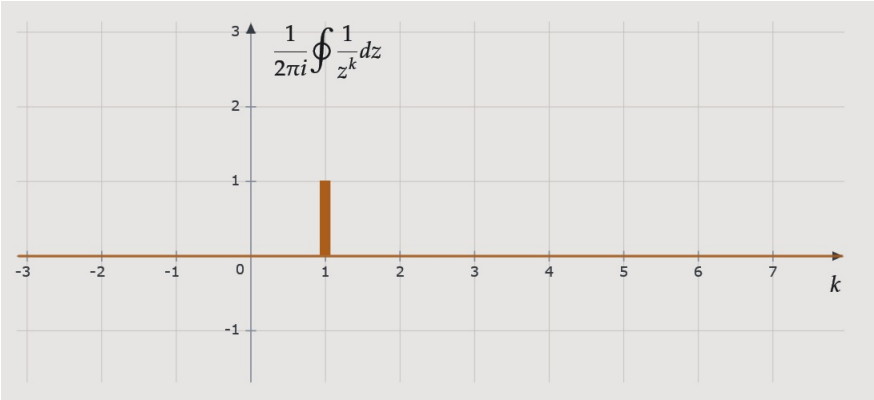

Figure 1: A graph of the value of the closed contour of $\frac{1}{2\pi i}\oint_C \frac{1}{z^k}$ as a function of k.

2 If not the singularity, then what?

We can essentially summarize exactly what is going on as follows:

In the contour integral, the phase of dz rotates around, and the case $k = 1$ corresponds uniquely to the case that exactly cancels out the phase rotation of dz. One on side of $k = 1$, the phase of $\frac{1}{z^k}dz$ will still rotate around in one direction, and on the other side of $k = 1$ the phase will rotate around in the other direction. The phase rotation is what causes the integral to vanish. Only $k = 1$ corresponds to the case where there is no (net) phase rotation, and all the integral contributions combine to some nonzero value.

In this way, the key point is that since $\frac{1}{z}$ has phase $e^{-i\theta}$, it will rotate around at the opposite rate which dz rotates around (since it dz still rotates like $e^{i\theta}$). As a result, equation (1) actually has much more to do with the fact that

$$\oint_{C_r} \bar{z}dz = 2\pi i r^2, \tag{3}$$

where $\bar{z}$ denotes the complex conjugate of z and C_r denotes a circle of radius r. Both $1/z$ and $\bar{z}$ have the opposite phase as z and cancel out the phase rotation of dz. This is the rough intuition. In some sense, we could stop here. However, there are ways in which we can make this more concrete, both illustrating exactly why there is a factor of i in equation (1) and also elaborating intuitively why this idea of phase rotation canceling should hold for all contour curves instead of just circles. We do that now.

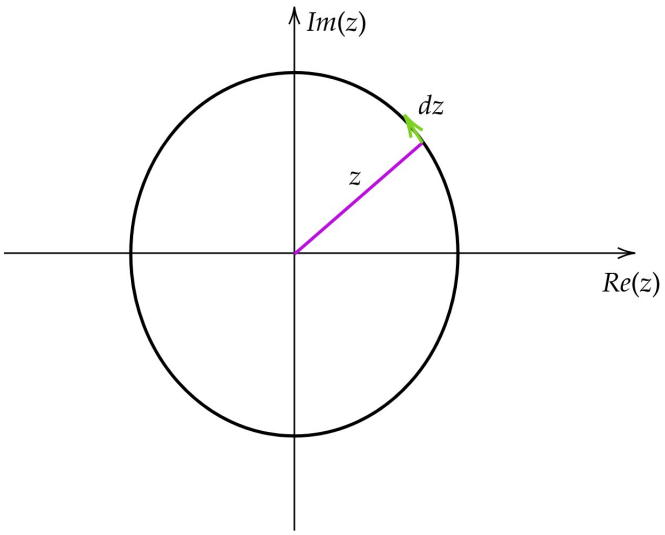

Figure 2: A graph of the value of the closed contour of $\oint_C \frac{1}{z^k}$ as a function of k.

3 Visualizing the case of the circle

The case of the circle simplifies things tremendously. Figure 2 illustrates a circular contour.

Positive contours imply moving counterclockwise. Thus, dz is (up to a real scalar) iz. This is where the i in $2\pi i$ comes from. Essentially, while dz rotates at the same rate as z, it starts out with a phase i times the phase of z and stays that way. Specifically,

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} ie^{i\theta} r d\theta, \tag{4}$$

where $ie^{i\theta}$ represents the phase of dz and $rd\theta$ represents it's magnitude. We also see concretely the idea that winding leads to cancellation:

$$\oint_C \frac{1}{z^k} dz = \int_0^{2\pi} \frac{1}{r^{k-1}} e^{-i(k-1)\theta} i d\theta = \frac{i}{r^{k-1}} \frac{-1}{i(k-1)} e^{-i(k-1)\theta}\Big|_0^{2\pi} = 0. \tag{5}$$

Equation (5) holds in the case k is an integer not equal to 1. The non integer case also vanishes but is beyond the point of this article. The fundamental point is that $\frac{1}{z^k} dz$ for $k \neq 1$ ends up winding around the circle. Because it is a circle, it winds at a constant rate, and around the whole circle it makes a full rotation and winds around to zero. We could stop here, now having established that the factor of i comes from dz being a rotation of z and that the $k \neq 1$ cases correspond to a net winding around. However, the geometry of a circle makes everything

simpler (almost too simple), and one might wonder if the same principles would still hold for more generic curves. For example, while $\frac{1}{z^k}dz$ might be winding around, the amount by which it winds around may vary at different parts of an arbitrary curve. There is a vague notion that for a full loop these different parts will conspire to yield zero and that variations from the circular case will combine to cancel (a sort of "what goes up must come down" effect), but it is beneficial to make this more clear. We do this now.

4 Why this generalizes to all generic curves

We first get out of the way the standard way to generalize the circular case to generic curves. After, we deal with a more intuitive picture for why perturbations of a circle don't change the contour integral of general functions (which are analytic in the region of perturbation).

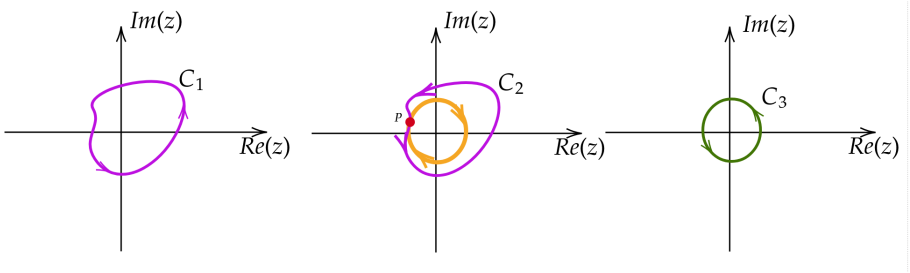

Figure 3: A graph of three contours which denote an arbitrary curve C_1, a circular curve C_3 which intersects the arbitrary curve at point P, and the curve $C_2 = C_1 - C_3$.

Figure 3 depicts an arbitrary curve C_1 and a circular curve C_3 which intersects C_1 at point P. For any simple curve C_1, we can construct a C_3 in this fashion. Both C_1 and C_3 are positively oriented, as they go around counterclockwise. Figure 3 also depicts C_2, which involves going the opposite direction around C_3, from P to P, then going positively around C_1. We see then, that

$$\oint_{C_1} \frac{1}{z^k}dz = \oint_{C_2} \frac{1}{z^k}dz + \oint_{C_3} \frac{1}{z^k}dz. \tag{6}$$

Now, C_2 is a closed contour, and $\frac{1}{z^k}$ is analytic in the region it encloses, since it does not enclose the origin. Thus, by Cauchy's theorem,

$$\oint_{C_2} \frac{1}{z^k}dz = 0. \tag{7}$$

Thus,

$$\oint_{C_1} \frac{1}{z^k}dz = \oint_{C_3} \frac{1}{z^k}dz, \tag{8}$$

demonstrating that we are allowed to generalize our results for circular curve to more general curves.

Now we move to a more intuitive picture for why contour integrals of circular curves are the same as for more generic curves. Specifically, we show that if one perturbs a circular curve, that does not change the contour integral of any general function, so long as that function is analytic in the region of perturbation. We do this by discretizing the curve.

Figure 4 depicts an arbitrary curve that is composed of discrete line elements. We shift point P by a value $\epsilon\Delta z$ and the draw the two shifted line elements in a new color. We use ϵ to keep track of the first order terms, and will neglect all terms of order $\mathcal{O}(\epsilon^2)$.

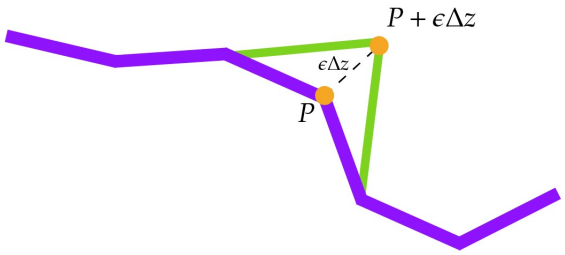

Figure 4: A sample curve comprised of discrete line elements with a perturbation of value $\epsilon\Delta z$ at point P.

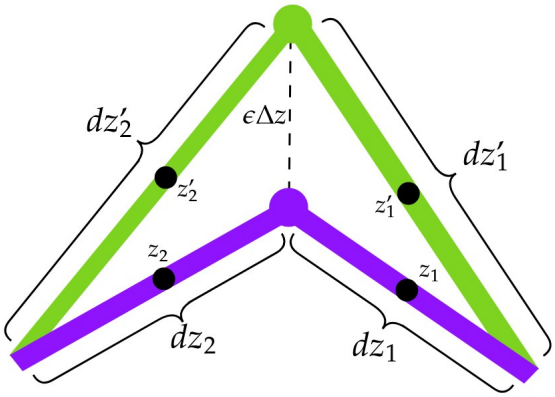

Figure 5: Figure 4 zoomed in at the point of perturbation

Figure 5 depicts Figure 4 zoomed in at the point of perturbation. The quantities dz_1 and dz_2 represent the original discrete line segments, while dz_1'

and dz_2' represent the new, perturbed quantities. For the line integral $\oint f(z)dz$, we will take $f(z)$ to be evaluated at the midpoint of each discrete dz, which is the most intuitive choice. The following argument also works if one takes the point of evaluation to be at an endpoint. Since we are simply considering first order terms, it does not make a difference which selection we choose. We denote the midpoints of the line segments and their shifted counterparts, using z_1 to denote the midpoint of dz_1 and so forth.

Now, if

$$S = \oint f(z)dz, \tag{9}$$

We'll denote the contribution to the integral from just the two original line segments dz_1 and dz_2 as

$$\delta S = f(z_1)dz_1 + f(z_2)dz_2. \tag{10}$$

Now, after the perturbation, we have

$$\delta S' = f(z_1')dz_1' + f(z_2')dz_2'. \tag{11}$$

Complex numbers add like vectors, so

$$dz_1' = dz_1 + \epsilon\Delta z,$$

while

$$dz_2' = dz_1 - \epsilon\Delta z.$$

From Figure 5, we see visually

$$z_1' = z_1 - \frac{dz_1}{2} + \frac{dz_1'}{2} = z_1 + \frac{\epsilon\Delta z}{2}.$$

We similarly deduce,

$$z_2' = z_2 + \frac{\epsilon\Delta}{2}.$$

Inputting into equation 11

$$\delta S' = f(z_1 + \frac{\epsilon\Delta z}{2})(dz_1 + \epsilon\Delta z) + f(z_2 + \frac{\epsilon\Delta z}{2})(dz_2 - \epsilon\Delta z).$$

Taylor expanding $f(z)$ to first order,

$$\delta S' = (f(z_1) + \frac{\epsilon\Delta z}{2}f'(z_1))(dz_1 + \epsilon\Delta z) + (f(z_2) + \frac{\epsilon\Delta z}{2}f'(z_2))(dz_2 - \epsilon\Delta z)$$

Ignoring $\mathcal{O}(\epsilon^2)$ terms,

$$\delta S' - \delta S = \epsilon\Delta z(f(z_2) - f(z_1)) - \frac{\epsilon\Delta z}{2}(f'(z_1)dz_1 + f'(z_2)dz_2).$$

However, we see from Figure 5 that

$$f(z_2) = f(z_1) + f'(z_1)\frac{dz_1}{2} + f'(z_2)\frac{dz_2}{2}. \tag{12}$$

Thus, we can conclude

$$\delta S' = \delta S, \tag{13}$$

and therefore the perturbation has no first order change to the line integral. Let's take a step back to analyze what was actually going on and exactly why the perturbation didn't change the line integral. First, we must recognize that our argument hinged on there being sensible notions of the derivative of $f(z)$ in the region of the perturbation. In other words, it necessitated $f(z)$ to be analytic in the region of perturbation (a necessary condition, since the closed loop integral of $\frac{1}{z}$ is not independent of perturbation and varies whether the loop encloses the origin or not).

On a more intuitive level, when we shift the curve by $\epsilon\Delta z$, dz_1 increases by that amount and dz_2 decreases by that same amount. If $f(z)$ is not changing in this region, it's clear that these shifts will cancel each other out. However, if $f(z)$ does change, then the shifts will not perfectly cancel, and there will be some remainder. This remainder, however, is canceled by the fact that we're not evaluating f at z_1 but at z_1' (and similarly for z_2). The whole idea is that, by being analytic, f has the same derivative in any direction. Thus, the derivative as one goes along the curve (in Figure 5 effectively going from right to left) is the same derivative as one goes up and down. This is the critical condition for being analytic as encoded in the Cauchy-Riemann equations, and it is what allows for the cancellation. We alluded earlier to a "what goes up must come down effect" denoting the idea that perturbations to the curve lead to changes that offset each other in the total integral. We now see that this occurs not globally, but locally, and it is not specific to functions of the form $\frac{1}{z^k}$. We analyzed just the first order terms, because we constructed the curve simply from line segments, and higher order terms would require greater precision in choices we made, such as putting the evaluation point at the midpoint of the line segment. The more precise explanation uses Cauchy's theorem which we gave the sketch of at the beginning of this section. The purpose of the discretizing of space is to give a more conceptual picture for exactly how the first order perturbation terms offset each other.

5 Conclusion

In this article, we uncovered the true mechanism that distinguishes $\frac{1}{z}$ from other functions of the form $\frac{1}{z^k}$ and leads to a nonzero closed integral around the origin. The nonzero value was not a direct result of $\frac{1}{z}$ blowing up at the origin, which is in fact true for all $\frac{1}{z^k}$ for $k > 0$. Rather, the mechanism was the fact that $\frac{1}{z}$ was the specific case in that broader class of functions that corresponded to canceling the phase rotation of dz. A net phase rotation led all the other integrals in that

class to wind around to zero, while the phase cancellation in the $\frac{1}{z}$ case led to steady accumulation of an imaginary quantity. We visually understood why the quantity had to be imaginary and found that the mechanism yielding a nonzero value for $\frac{1}{z}$ is similar to the nonzero value found when calculating the contour integral of the complex conjugate $\bar{z}$. However, as the example of the circle illuminates, unlike $\bar{z}$, $\frac{1}{z}$ also has a $\frac{1}{r}$ term which cancels the r implicitly encoded in $dz = rd\theta$, and therefore the magnitude of the contour integral is a constant.

We then explained why the analysis of the circle can be generalized to generic curves, first using the standard method of Cauchy's theorem, and then using a conceptual picture of discretizing the curve into line segments and perturbing one point in the curve. Such a conceptual picture illustrated the necessity of $f(z)$ being analytic in the region of perturbation, as it required the derivative of $f(z)$ to be the same in all directions, the defining quality of being analytic.